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This is a comment on the paper "Electron Mass Operator in a Strong Magnetic Field and Dynamical Chiral Symmetry Breaking" by A. V. Kuznetsov and N. V. Mikheev [Phys. Rev. Lett. 89 (2002) 011601]. We show that the main conclusions of the paper are incorrect.

Magnetic catalysis of chiral symmetry breaking [1] is a well established phenomenon in $(2+1)$ and $(3+1)$ dimensional relativistic models. The general result is that a constant magnetic field leads to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions. The essence of the effect is the dimensional reduction $D \to D-2$ in the dynamics of fermion pairing in a magnetic field [1].

The realization of this phenomenon in quantum electrodynamics (QED) was studied in detail in Refs. [2–8]. In particular, in Refs. [6,7], we derived an asymptotic expression for the fermion dynamical mass in the chiral limit in QED, reliable for a weak coupling. In a recent Letter [9], Kuznetsov and Mikheev attempted to revise that analysis. These authors claimed that:

(i) In a model with N charged fermions, a critical number N_{cr} exists for any value of the electromagnetic coupling constant, such that the dynamical mass does not arise for $N > N_{cr}$, and thus the chiral symmetry remains unbroken.

(ii) The fermion dynamical mass is generated with a double splitting for $N < N_{cr}$.

In this Comment, we will show that these conclusions of Ref. [9] are incorrect. In fact, we show that

(i′) The generation of the fermion dynamical mass takes place for any number of the fermion flavors N if the magnetic field is less than the value of the Landau pole, i.e., if the theory itself is well defined. The erroneous conclusion (i) of Ref. [9] follows from an inappropriate treatment of the Schwinger-Dyson (gap) equation for large N.

(ii′) There is a unique solution for the dynamical fermion mass in any given theory with a fixed value of N.

Let us start by mentioning that the gap equation (12) of Ref. [9] for the case of zero bare mass of fermions,

$$\frac{\alpha_R}{2\pi}\left(\ln\frac{\pi}{N\alpha_R}-\gamma_E\right)\ln\frac{|eB|}{m^2}=1, \tag{1}$$

is the simplest approximation to the gap equation previously derived in our papers [6,7] in the limit of a weak coupling and not too large values of N. Here, by definition, $m \equiv m_{dyn}(|eB|)$ is the dynamical mass and α_R is the coupling constant at the scale of $\sqrt{|eB|}$. One should note that the validity of the above equation breaks down when the number of fermions becomes large. Indeed, with the increasing value of N the photon mass squared $M_\gamma^2 = 2N\alpha_R|eB|/\pi$ should eventually become comparable to the magnetic field scale $|eB|$. When this happens the leading-log approximation used in Ref. [9] fails, and therefore one cannot trust equation (1) any longer. In fact, this could be seen from Eq. (1) itself: a simple-minded large N limit leads to the change of sign on the left hand side of the equation, suggesting that no reasonable solution for the dynamical mass exists. Obviously, this is incorrect because the original integral form of the gap equation [see Eq. (54) in Ref. [7]] does not share this unphysical property. It is not difficult to derive an approximate algebraic form of the gap equation which works reasonably well for all values of N. Its explicit form reads

$$-\frac{\alpha_R}{2\pi}\exp\left(\frac{N\alpha_R}{\pi}\right)\mathrm{Ei}\left(-\frac{N\alpha_R}{\pi}\right)\ln\frac{|eB|}{m^2}=1, \tag{2}$$

where $\mathrm{Ei}(z)$ is the exponential integral function. Notice, that this gap equation has a nontrivial solution for any number of fermions. In particular, in the limit of large N, the corresponding solution for the dynamical mass reads

$$m \simeq \sqrt{|eB|}\exp(-N), \quad \text{for} \quad N \gg \pi/\alpha_R. \tag{3}$$

This demonstrates that the conclusion of Ref. [9] about the existence of a critical value of N is incorrect.

For clarity, let us emphasize that the only "assumption" in the above derivation was that the scale of the magnetic field lies below the scale of the Landau pole (mathematically, this is expressed as $\alpha_R < \infty$). This condition is of course necessary because the theory is not well defined otherwise.

Another claim of Ref. [9] is that "a fermion dynamical mass is generated with a doublet splitting for $N < N_{cr}$". In fact, this conclusion was reached as a result of misinterpretation of the following feature of the gap equation (1): this equation has two solutions with different values of the fermion mass for each choice of $|eB|$, N and the coupling constant $\alpha \equiv \alpha(m)$ related to the renormalization group (RG) scale $\mu = m$. It is important that m is the *dynamical* mass, determined from the gap equation, and that in Ref. [9] the running coupling is taken from the one-loop RG equations in QED *without* a magnetic field. Let us show that the interpretation of this feature as a mass splitting is incorrect. To see this, we turn to the RG equation for the running coupling constant:

$$\alpha(m) = \frac{\alpha(\mu)}{1 - \alpha(\mu) b_0 \ln(m^2/\mu^2)}, \quad \text{with} \quad b_0 = \frac{N}{3\pi}. \tag{4}$$

Now, suppose that indeed, for given values of $|eB|$ and N, the gap equation can have two solutions with different masses, m_1 and m_2, and that the constraint $\alpha(m_1) = \alpha(m_2)$ can be satisfied at the same time. Then, it is easy to see that these two solutions correspond not to the same theory, as the authors of Ref. [9] believe, but to two different theories. Indeed, since $\alpha(m_1) = \alpha(m_2) \equiv \alpha$, the values of the two corresponding RG coupling constants $\alpha_1(\mu)$ and $\alpha_2(\mu)$ related to the *same* scale μ are different. The latter directly follows from equation (4):

$$\alpha_i(\mu) = \frac{\alpha}{1 + \alpha\, b_0 \ln(m_i^2/\mu^2)}, \qquad i = 1, 2. \tag{5}$$

Therefore, the solutions m_1 and m_2 correspond to two theories with different RG coupling constants. In those theories, in particular, the coupling constants take different values at the scale $\mu = \sqrt{|eB|}$. This in turn implies that there is only one solution for the dynamical fermion mass in any given theory with a fixed value of N.

We emphasize that there is nothing wrong with the parameterization of a solution in QED in a magnetic field by a coupling constant in QED without the magnetic field. One can use a coupling constant related to any fixed scale μ. The choice of the scale $\mu = m$ in Ref. [9], although possible, is contrived: this scale can be determined self-consistently only after solving the gap equation, and therefore it is not a free parameter. This subtlety led the authors to the incorrect interpretation regarding the appearance of the "mass splitting" [10].

In connection with this, we would like to point out that the claim in Ref. [9] that the coupling constant renormalization was not taken into account in Refs. [6,7] is without basis. Along with the polarization effects, the renormalization of the coupling constant was taken properly into account in Refs. [6,7]. In fact, the choice of the coupling constant α_R related to the RG scale $\mu = \sqrt{|eB|}$ in Refs. [6,7] is of course nothing else but a coupling constant renormalization. This choice is the most convenient (although not unique) parameterization of the gap equation and its solution. Indeed, $\sqrt{|eB|}$ is the only free dimensional parameter in this problem [11].

In conclusion, the magnetic catalysis of chiral symmetry breaking is always realized as soon as the theory itself is well defined. In the case of QED, this simply requires that the values of magnetic fields should lie below the scale of the Landau pole.

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[10] It is appropriate to mention that even the existence of two solutions with $\alpha(m_1) = \alpha(m_2)$ is no longer valid after the corrected gap equation (2) is considered.
[11] Another advantage of this choice is that the value of the coupling related to the scale $\sqrt{|eB|}$ (as well as to any scale above it) is essentially the same in QED with and without the magnetic field [7].